Analysis I Lecture 16

Last time

- Algebra of limits for functions
- Infinite limits: $\lim_{x \to \pm\infty} f(x)$

  or $\lim_{x \to x_0} f(x) = \pm\infty$
- Algebra of infinite limits.

Statements about sequences can be translated to statements about functions

- Continuous functions:

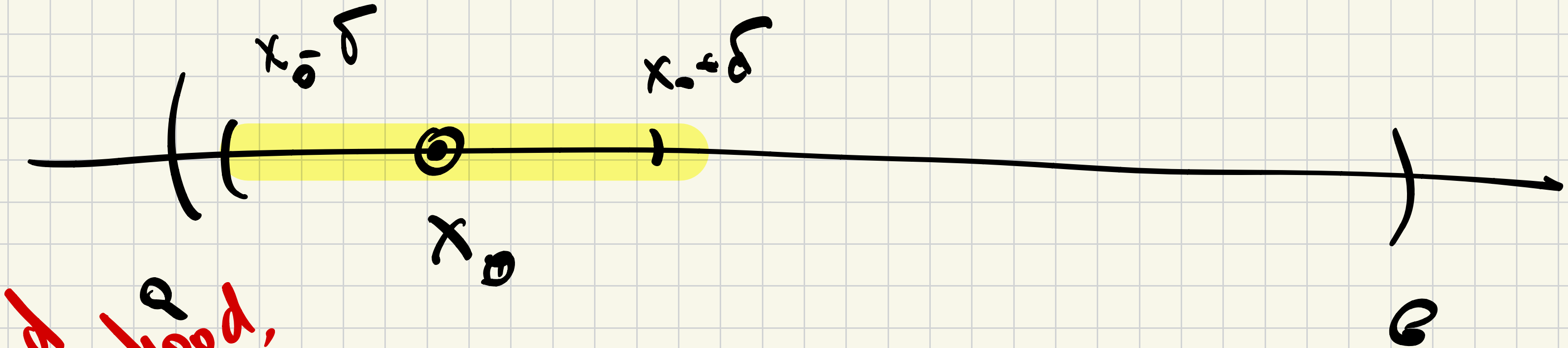
- $f(x)$ is continuous at x_0 if $\lim_{x \to x_0} f(x)$ exists and equals to $f(x_0)$.

- $f(x)$ is continuous on $\mathcal{U}$ if it is continuous at every $x_0 \in \mathcal{U}$.

technical note: $\mathcal{U}$ should contain punctured neighborhood of every $x_0 \in \mathcal{U}$

For instance $U = (a, b)$ then

$\forall \; x_0 \in (a,b)$ we can consider



punctured neighborhood,

$(x_0-\delta, \; x_0+\delta) \setminus \{x_0\} \subset U$ as long

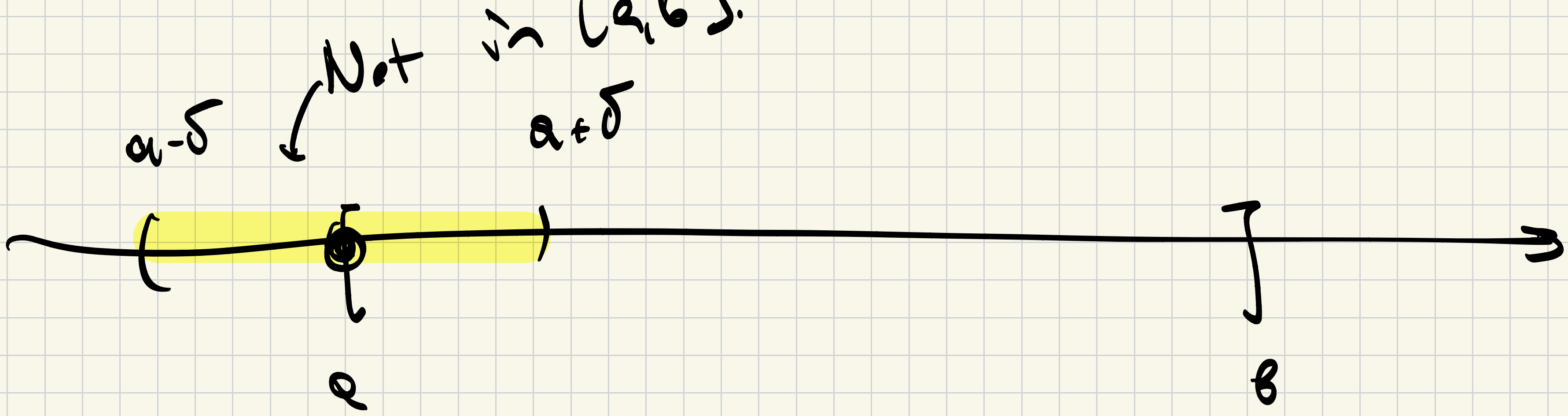
as $x_0 - \delta > a$, $x_0 + \delta < b$.

But if $\mathcal{U} = [a, b]$

then $a \in \mathcal{U}$ but there is

no punctured neighborhood of a

contained in $\mathcal{U}$:



- Composition of functions:

$$g \circ f(x) := g(f(x))$$

- Chance of variables (continuity)

Let f, g be two functions and assume $g \circ f$ is defined then

1) f is continuous at x_0
2) g is continuous at $f(x_0) = y_0$

$\Rightarrow$ $g \circ f$ cont. at x_0.

Change of variables (limits)

Let f, g be two functions and assume $g \circ f$ is defined then if

1) $\lim_{x \to x_0} f(x) = y_0$ 2) $\lim_{y \to y_0} g(y) = \ell$

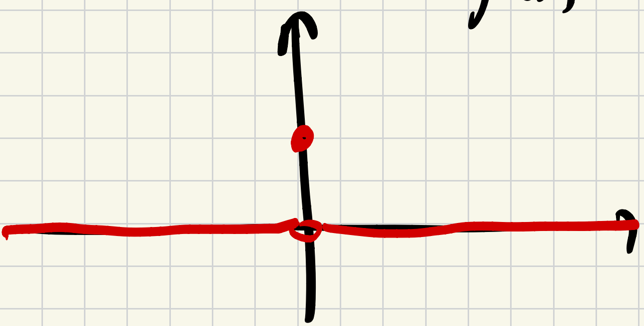
3) $\exists$ a punctured neighborhood of x_0 s.t. for every x in it $f(x) \neq y_0$ } technical condition ...

Then $\lim_{x \to x_0} (g \circ f)(x) = \ell$

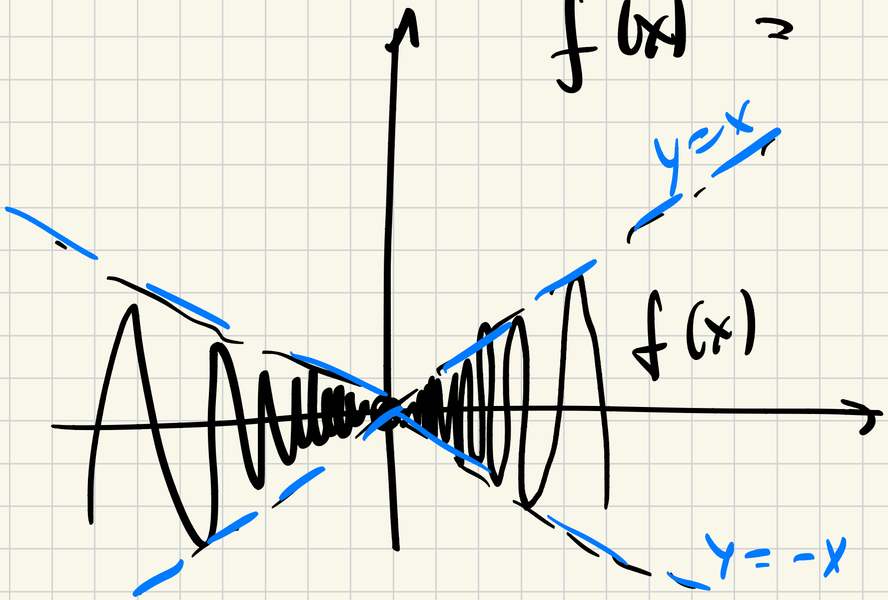
Why condition 3 is there?

Example $g(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$g(x)$



$f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$



Notice that for any $\varepsilon > 0$ $\exists$ $|x| < \varepsilon$ s.t. $f(x) = 0$.

We have

- $\underbrace{\lim_{x \to \infty} f(x) = 0}_{\text{use squeeze thm and inequalities}}$

$$-x \leq x \cdot \sin(x) \leq x$$

- $\lim_{x \to \infty} g(x) = 0$

Naive change of variables tells that

$$\lim_{x\to 0} g\circ f = \lim_{y \to \underbrace{\lim_{x\to 0} f(x)}_{=0}} g(y) =$$

$$= \lim_{y\to 0} g(y) = 0.$$ But this is wrong!!! Because part 3 of proposition is not satisfied

In fact $\lim_{x \to 0} g \circ f(x)$ does not exists:

Consider

We have $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$

lets compute $g \circ f(a_n)$ and $g \circ f(b_n)$

$a_n = \frac{1}{\pi \cdot n} \qquad b_n = \frac{1}{\pi \cdot n + \frac{\pi}{2}} \qquad \begin{array}{l} a_n \neq 0 \\ b_n \neq 0 \end{array}$

$$g \circ f(a_n) = g\big(f(a_n)\big) =$$

$$= g\left(a_n \cdot \sin\left(\frac{1}{a_n}\right)\right) = g\left(\frac{\overbrace{\sin(\pi \cdot n)}^{0 \ \forall n}}{\pi \cdot n}\right) =$$

$$= g(0) = 1$$

$$\Rightarrow \boxed{\lim_{n \to \infty} g \circ f(a_n) = 1}$$

For b_n:

$$\lim_{n\to\infty} g(f(b_n)) = \lim_{n\to\infty} g\left(b_n \cdot \sin\left(\frac{1}{b_n}\right)\right)$$

$$= \lim_{n\to\infty} g\left(\frac{\overbrace{\sin\left(\pi n + \frac{\pi}{2}\right)}^{\neq 0 \ \forall n}}{\pi n + \frac{\pi}{2}}\right) = \lim_{n\to\infty} 0 = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty} g(f(b_n)) = 0}$$

So we get

$$\lim_{n\to\infty} g\circ f(a_n) = 1 \neq 0 = \lim_{n\to\infty} g\circ f(b_n)$$

So $\boxed{\lim_{x\to 0} g\circ f \text{ does not exists.}}$

Reason is that there is no neighborhood of 0 s.t. $\forall x$ in it $f(x)\neq 0$.

Today:

- One side limits
- limits of monotone functions
- Left and right continuity.
- (Extension by continuity.)

One side limits.

Left Limit.

Definition: Let $f: E \to \mathbb{R}$ be a function

f is defined to the left of x_0

s.t. $(x_0-\delta, x_0] \subset E$ then we say that

the left limit of f at x_0 is ℓ

$\left(\lim_{x \to x_0^-} f(x) = \ell\right)$ if for any sequence

(a_n) with $\lim_{n\to\infty} a_n = x_0$ and $a_n < x_0$

we have $\lim_{n\to\infty} f(a_n) = \ell$.

Left limit

we don't care if f is defined for $x \geq x_0$.

$$\lim_{n\to\infty} f(a_n) = \ell$$

Right limit

Definition: Let $f: E \to \mathbb{R}$ be a function f defined to the right of x_0 s.t. $(x_0, x_0+\delta) \subseteq E$, then we say that the Right limit of f at x_0 is ℓ $\left(\lim_{x \to x_0^+} f(x) = \ell \right)$ if for any sequence (a_n) with $\lim_{n\to\infty} a_n = x_0$ and $a_n > x_0$ we have $\lim_{n\to\infty} f(a_n) = \ell$.

Examples

Let $f(x) = \sqrt{x}$ for $x \geq 0$

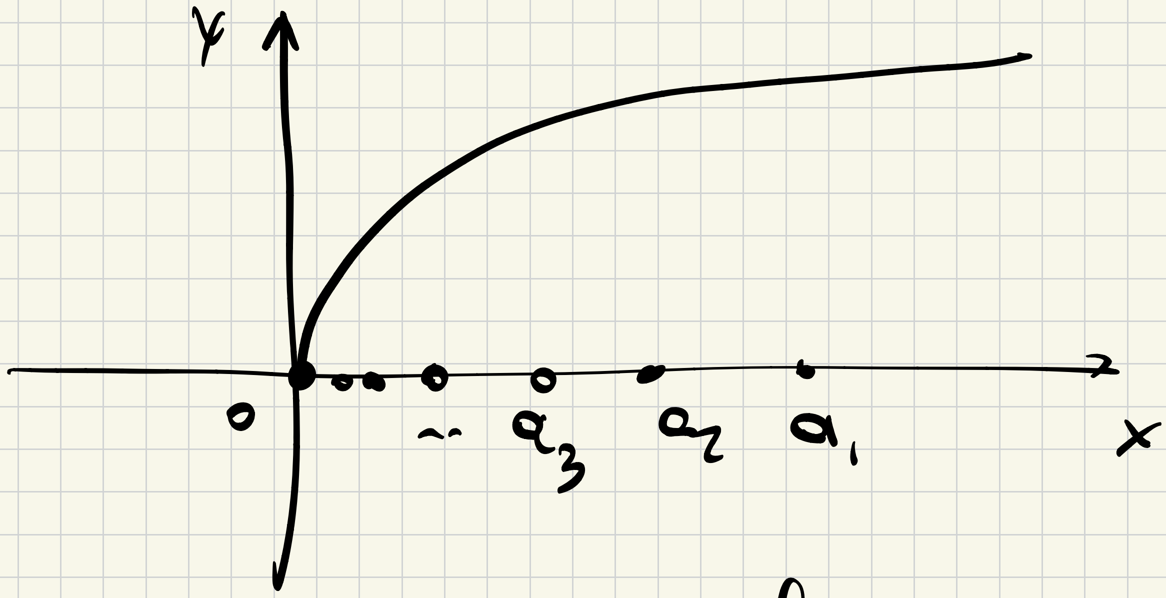
Notice $\lim_{x \to 0} \sqrt{x}$ is not defined

because $\sqrt{x}$ is not defined for $x < 0$

But $\sqrt{x}$ is defined for $x \geq 0$

So we ask what is the right limit at 0?

$f(x) = \sqrt{x}$



- Need to find $\lim_{n\to\infty} \sqrt{a_n}$ for every (a_n) s.t. $\lim_{n\to\infty} a_n = 0$ and $a_n > 0$.

<u>Claim</u> Any such $\lim_{n\to\infty} \sqrt{a_n} = 0$.

$$\Rightarrow \boxed{\lim_{x\to 0^+} \sqrt{x} = 0.}$$

Proof of the claim

Let $(a_n) \to 0$ for $n \to \infty$ and $a_n \geq 0$

then $\forall \varepsilon \ \exists N$ s.t. $a_n < \varepsilon^2$

for $n > N$

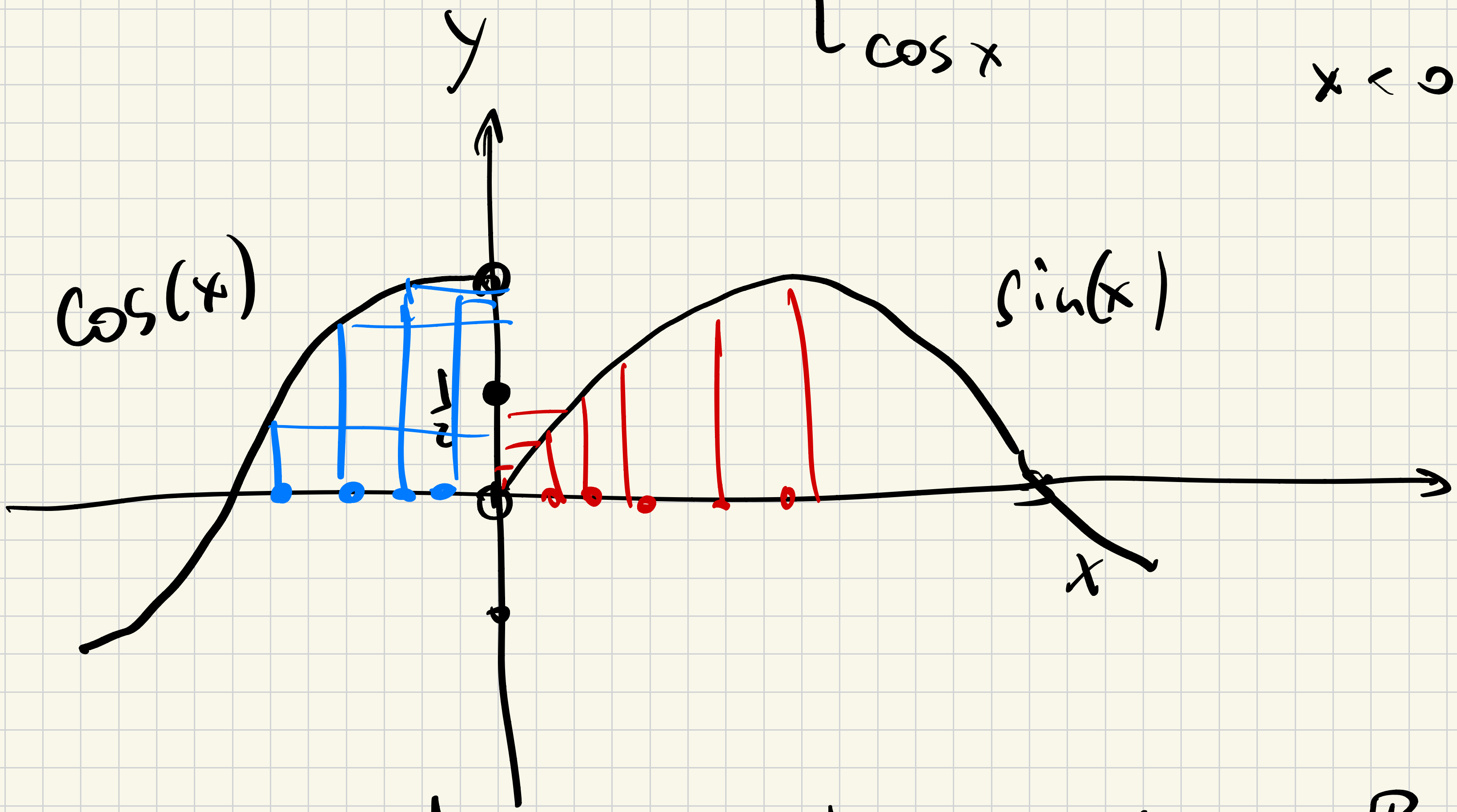
$$\Rightarrow \sqrt{a_n} < \sqrt{\varepsilon^2} = \varepsilon \Rightarrow$$

$$\Rightarrow \lim_{n \to \infty} \sqrt{a_n} = 0 .$$

$\blacksquare$

Example

$$f(x) = \begin{cases} \sin x & x > 0 \\ \frac{1}{2} & x = 0 \\ \cos x & x < 0 \end{cases}$$



f is defined on the whole $\mathbb{R}$

So we ask for $\lim_{x \to 0} f(x)$?

$\lim_{x \to 0^-} f(x)$ or $\lim_{x \to 0^+} f(x)$!

since $f(x) = \cos(x)$ for $x < 0$

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x) = 1$

since $f(x) = \sin(x)$ for $x > 0$

- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin(x) = 0$

Finally

$\lim_{x \to 0} f(x)$ does not exist:

Indeed,
$$\lim_{n \to \infty} f\left(\frac{1}{n}\right) = 0$$
$$\neq$$
$$\lim_{n \to \infty} f\left(-\frac{1}{n}\right) = 1$$

■

Proposition 1.42 Let $f: E \to \mathbb{R}$ be a function defined in a punctured neighborhood of x_0 and s.t.

$$\lim_{x \to x_0^-} f(x) = \ell_1 \quad , \quad \lim_{x \to x_0^+} f(x) = \ell_2$$

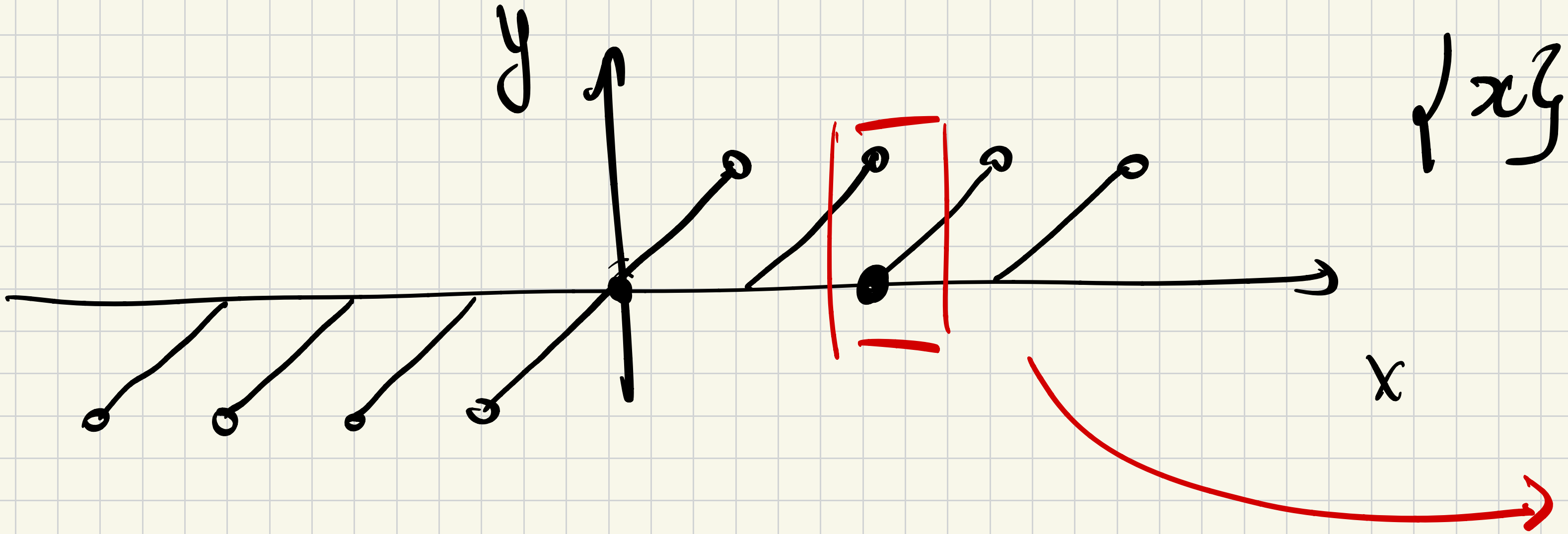
then:

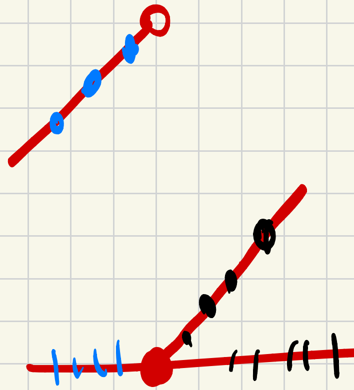
$$\lim_{x \to x_0} f(x) = \ell \iff \ell_1 = \ell_2 = \ell$$

($\Longrightarrow$ In example before)

Example

Consider $f(x) = \{x\}$





Let's focus for $x \geq 0$:

If $x_0 \in \mathbb{N}_{>0}$ then

$$\lim_{x \to x_0^-} \{x\} = 1$$

$$\lim_{x \to x_0^+} \{x\} = 0$$

$\Rightarrow \lim_{x \to x_0} \{x\}$ does not exists.

On the other hand if

$x_0 \notin \mathbb{N}$ then $\lim\limits_{x \to x_0} \{x\} = \{x_0\}$

$\Rightarrow$ both left and right limit exist and equal to $\{x_0\}$

$$\lim_{x \to x_0^-} \{x\} = \{x_0\} = \lim_{x \to x_0^+} \{x\}.$$

Limits of monotone functions

Similar to sequences we can guarantee existence of (left / right) limits for monotone functions.

Prop. 1.45 in notes on monotone functions

Proposition Let f be a monotone function on $E \ni x_0$.

1) If f is defined to the left of x_0

$\lim_{x \to x_0^-} f(x)$ exists

2) If f is defined to the right of x_0

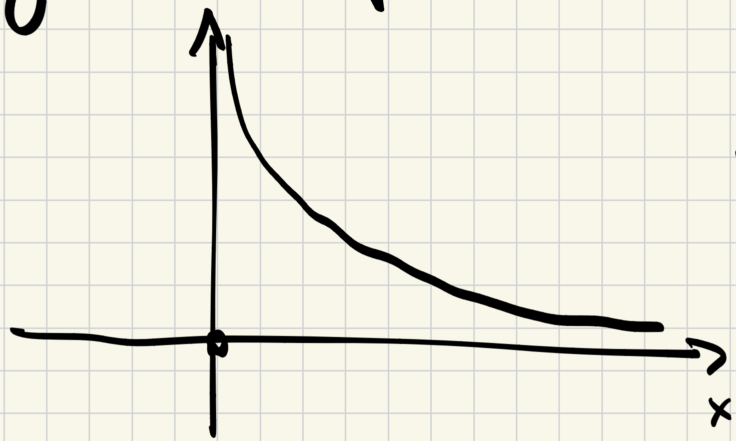
$\lim_{x \to x_0^+} f(x)$ exists

3) If f is defined in a $\underbrace{\text{neighborhood of } \pm\infty}_{\text{i.e } (C, +\infty) \text{ or } (-\infty, C)}$

then $\lim_{x \to \pm\infty} f(x)$ exists.

In this proposition left/right limits might be <u>infinite</u>!

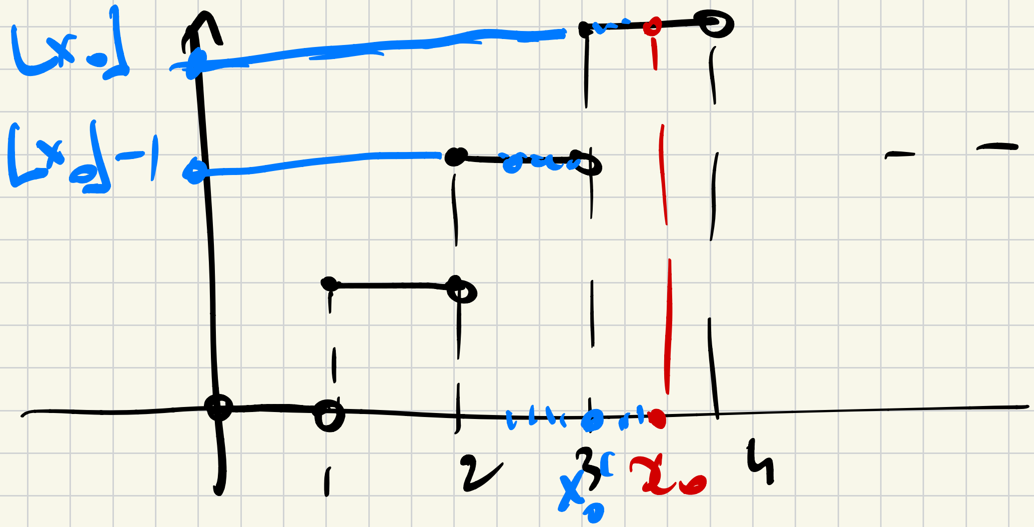
e.g $f(x) = \frac{1}{x}$ for $x > 0$



$f(x)$ is monotone, defined to the right of 0 and $\lim_{x \to 0^+} f(x) = +\infty$

Example

$f(x) = \lfloor x \rfloor$ for simplicity take $x_0 > 0$



$f(x)$ is monotone $\Rightarrow \forall x_0 > 0$

both $\lim\limits_{x \to x_0^-} \lfloor x \rfloor$ and $\lim\limits_{x \to x_0^+} \lfloor x \rfloor$ exist

Indeed, If $x_0 \notin \mathbb{N}$

$$\lim_{x \to x_0^+} \lfloor x \rfloor = \lim_{x \to x_0^-} \lfloor x \rfloor = \lfloor x_0 \rfloor$$

If $x_0 \in \mathbb{N}_{>0}$ then

$$\lim_{x \to x_0^+} \lfloor x \rfloor = \lfloor x_0 \rfloor \qquad \lim_{x \to x_0^-} \lfloor x \rfloor = \lfloor x_0 \rfloor - 1$$

$\blacksquare$

Left and right continuity

Definition Let $f : E \to \mathbb{R}$ and $x_0 \in E$

we say that

$f(x)$ is Left continuous at x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$

$f(x)$ is Right continuous at x_0 if $\lim_{x \to x_0^+} f(x) = f(x_0)$

Example:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ x - 1 & \text{if } x < 0 \end{cases}$$